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Approximation of Entire Functions over Bounded Domains

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1. INTRODUCTION

Let C be a Jordan curve, D its interior and E its exterior. For $1 \leqslant p \leqslant \infty$, let $L^p(D)$ denote the set of functions f holomorphic in D and such that

$$\|f\|_{L^p(D)} = \left(\frac{1}{A} \iint_D |f(z)|^p \, dx \, dy\right)^{1/p} < \infty$$

where A is the area (Lebesgue measure) of D. (If $p = \infty$, the last inequality is to be understood as $\sup_D |f| < \infty$). For $f \in L^p(D)$, set

$$E_n{}^p = E_n{}^p(f; D) = \min_{\pi_n} \|f - \pi_n\|_{L^p(D)}$$

where π_n is an arbitrary polynomial of degree at most n. The purpose of this note is to prove the following result.

THEOREM. *Let $2 \leqslant p \leqslant \infty$. A function $f \in L^p(D)$ is the restriction to D of an entire function if and only if*

$$\lim_{n \to \infty} (E_n{}^p)^{1/n} = 0.$$

In that case, f is of finite order ρ if and only if

$$\limsup_{n\to\infty} \frac{n \log n}{-\log E_n{}^p} = \rho \tag{1}$$

and, if $\rho > 0$, of finite type σ if and only if

$$\limsup_{n\to\infty} \frac{n}{e\rho} (E_n{}^p)^{\rho/n} = \sigma d^\rho \tag{2}$$

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where d is the transfinite diameter of C. In the case $p = \infty$, it is enough to assume that f is continuous on D.

In the case $p = 2$ and $D = \{z : |z| < 1\}$, this theorem was already proved by A.R. Reddy [6]. In the case $p = \infty$ a proof of the theorem was given by A.V. Batyrev [8]. For the sake of completeness and since it is short, we shall nevertheless include proof of this last case. This case is reminiscent of a classical result of S. Bernstein [1, p. 113] on the approximation of continuous functions on finite intervals, a result that was recently extended by R.S. Varga [7].

2. The Faber Series

Since an arbitrary Jordan curve can be approximated from the inside as well as from the outside by analytic Jordan curves (using Riemann's mapping theorem, for example), since $E_n{}^p(f; D)$ increases with D and since the transfinite diamater is a continuous set function (see M. Fekete [5]), it is enough to prove the theorem for analytic Jordan curves. Similarly, if f is continuous in D and if $\lim_{n\to\infty} E_n{}^\infty = 0$, then, by virtue of Morera's theorem, f is holomorphic in D. So let f be holomorphic inside the analytic Jordan curve C.

Let us recall a few facts about Faber polynomials. Let $w = \varphi(z)$ map E conformally onto $\{w : |w| > 1\}$ in such a way that $\varphi(\infty) = \infty$ and that $\varphi'(\infty) > 0$. For $|z|$ sufficiently large, one has

$$\varphi(z) = \frac{z}{d} + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots \tag{3}$$

and since C is assumed to be analytic φ is actually holomorphic on C also. The nth Faber polynomial $F_n(z)$ of C is the principal part at ∞ of $(\varphi(z))^n$, so that

$$F_n(z) = \frac{z^n}{d^n} + \cdots .$$

These polynomials where introduced by G. Faber who proved [4] that

$$F_n(z) \sim (\varphi(z))^n \text{ as } n \to \infty \tag{4}$$

uniformly for $z \in E$ and that

$$\lim_{n\to\infty} (\max_{z\in C} |F_n(z)|)^{1/n} = 1. \tag{5}$$

Any function f holomorphic in D can be represented by its Faber series

$$f(z) = \sum_{n=0}^{\infty} a_n F_n(z) \tag{6}$$

where

$$a_n = \frac{1}{2\pi i} \int_{|w|=r} f(\varphi^{-1}(w))\, w^{-(n+1)}\, dw$$

and $r < 1$ is sufficiently close to 1 so that φ^{-1} is holomorphic and univalent in $|w| \geqslant r$, the series converging uniformly on compact subsets of D. We shall need the following result.

LEMMA 1. *The function f is the restriction to D of an entire function if and only if*

$$\lim_{n\to\infty} (|a_n|)^{1/n} = 0. \tag{7}$$

In that case, it is of finite order ρ if and only if

$$\limsup_{n\to\infty} \frac{n \log n}{-\log |a_n|} = \rho \tag{8}$$

and, if $\rho > 0$, of finite type σ if and only if

$$\limsup_{n\to\infty} \frac{n}{e\rho} |a_n|^{\rho/n} = \sigma d^\rho.$$

The proof of this theorem is analogous the proof of the classical result for the Taylor coefficients of an entire function (see R. P. Boas [2, p. 9]). Equation (7) is already contained in [4]. Let us sketch a proof of (8). Assuming f entire of finite order ρ, one can write

$$a_n = \frac{1}{2\pi i} \int_{|w|=R} f(\varphi^{-1}(w))\, w^{-(n+1)}\, dw$$

with arbitrarily large R. Since, by (3),

$$\lim_{|w|\to\infty} \frac{\varphi^{-1}(w)}{w} = d$$

one has for all sufficiently large $|w|$ that

$$(d - \epsilon)|w| \leqslant |\varphi^{-1}(w)| \leqslant (d + \epsilon)|w|$$

so that

$$|f(\varphi^{-1}(w))| \leqslant e^{((d+\epsilon)|w|)^{\rho+\epsilon}}$$

and that

$$|a_n| \leqslant R^{-n} e^{((d+\epsilon)R)^{\rho+\epsilon}}$$

for all R sufficiently large. To minimize the right member of this inequality, we select

$$R = \frac{1}{d+\epsilon}\left(\frac{n}{\rho+\epsilon}\right)^{1/(\rho+\epsilon)}$$

and obtain

$$\limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|} \leqslant \rho.$$

On the other hand, if

$$|a_n| \leqslant Kn^{-n/(\rho+\epsilon)}$$

(here and in the sequel K is a number independent of n, not necessarily the same in each occurence), the formula (6) will hold for any z and, hence,

$$|f(z)| \leqslant K\sum_{n=0}^{\infty} n^{-n/(\rho+\epsilon)}\,|F_n(z)|.$$

Using (4), we get that

$$|F_n(z)| \leqslant K\,|\varphi(z)|^n$$

for all $z \in E$ and, using (3), that

$$|\varphi(z)| \leqslant \frac{|z|}{d-\epsilon} \tag{9}$$

for all sufficiently large $|z|$. These inequalities imply that

$$|f(z)| \leqslant K\sum_{n=0}^{\infty} n^{-n/(\rho+\epsilon)}\left(\frac{|z|}{d-\epsilon}\right)^n \tag{10}$$

for all z with sufficiently large modulus. Since the largest term of the series $\sum_n (x/n^\lambda)^n$ is $e^{(\lambda/e)x^{1/\lambda}}$ and it occurs for $n = (1/e)\,x^{1/\lambda}$, a standard argument shows that (10) implies

$$|f(z)| \leqslant Ke\left(\frac{|z|}{d-\epsilon}\right)^{\rho+2\epsilon}$$

that is, f is of finite order at most ρ.

3. The Case $p = 2$

Consider the polynomials

$$p_n(z) = \lambda_n z^n + \cdots\ (\lambda_n > 0)$$

defined through

$$\frac{1}{A}\iint_D p_n(z)\,\overline{p_m(z)}\,dx\,dy = \delta_{n,m}\,.$$

These polynomials were first considered by T. Carleman [3] who proved that

$$p_n(z) \sim \left(\frac{(n+1)\,A}{\pi}\right)^{1/2} \varphi'(z)(\varphi(z))^n \qquad \text{as} \quad n \to \infty \tag{11}$$

uniformly for $z \in E$. Any function $f \in L^2(D)$ can be expanded in these polynomials

$$f(z) = \sum_{n=0}^{\infty} b_n p_n(z) \tag{12}$$

where

$$b_n = \frac{1}{A}\iint_D f(z)\,\overline{p_n(z)}\,dx\,dy$$

and the series converges uniformly on compact subsets of D.

Parseval's relation yields

$$E_n^2 = \left(\sum_{k=n+1}^{\infty} |\, b_k \,|^2\right)^{1/2}.$$

We shall use the following elementary lemma.

LEMMA 2.

$$\limsup_{n\to\infty} (E_n^2)^{1/n} = \limsup_{n\to\infty} |\, b_n \,|^{1/n},$$

$$\limsup_{n\to\infty} \frac{n \log n}{-\log E_n^2} = \limsup_{n\to\infty} \frac{n \log n}{-\log |\, b_n \,|}$$

and, for any $\rho > 0$,

$$\limsup_{n\to\infty} \frac{n}{e\rho} (E_n^2)^{\rho/n} = \limsup_{n\to\infty} \frac{n}{e\rho} |\, b_n \,|^{\rho/n}. \tag{13}$$

Let us, for instance, prove (13). Since

$$|\, b_{n+1} \,| \leqslant E_n^2\,,$$

the inequality

$$\limsup_{n\to\infty} \frac{n}{e\rho} |\, b_n \,|^{\rho/n} \leqslant \lim_{n\to\infty} \frac{n}{e\rho} (E_n^2)^{\rho/n}$$

is trivial. On the other hand, if

$$\limsup_{n\to\infty} \frac{n}{e\rho} | b_n |^{\rho/n} = \sigma$$

is finite, then

$$| b_n | \leqslant K \left(\frac{e\rho(\sigma + \epsilon)}{n} \right)^{n/\rho}$$

so that

$$\begin{aligned}(E_n^2)^2 &\leqslant K \sum_{k=n+1}^{\infty} \left(\frac{e\rho(\sigma + \epsilon)}{k} \right)^{2k/\rho} \\ &\leqslant K \sum_{k=n+1}^{\infty} \left(\frac{e\rho(\sigma + \epsilon)}{n+1} \right)^{2k/\rho} \\ &= K \left(\frac{e\rho(\sigma + \epsilon)}{n+1} \right)^{2(n+1)/\rho} \left(1 - \left(\frac{e\rho(\sigma + \epsilon)}{n+1} \right)^{2/\rho} \right)^{-1}\end{aligned}$$

for $n > 2e\rho(\sigma + \epsilon)$, say, and hence

$$\limsup_{n\to\infty} \frac{n}{e\rho} (E_n^2)^{\rho/n} \leqslant \sigma.$$

Let us now prove (2) in the case $p = 2$. We assure that f is an entire function of order $\rho > 0$ and type σ. Using lemma 1, we obtain for its Faber coefficients the estimate

$$| a_n | \leqslant K \left(\frac{e\rho d^\rho(\sigma + \epsilon)}{n} \right)^{n/\rho}.$$

Now, taking into account the orthonormality of the polynomials $p_n(z)$, one has

$$b_n = \sum_{k=n+1}^{\infty} a_k \frac{1}{A} \iint_D F_k(z) \overline{p_n(z)} \, dx \, dy$$

and hence

$$| b_n | \leqslant \sum_{k=n+1}^{\infty} | a_k | \max_{z\in C} | F_k(z)|.$$

Since, by (5)

$$\max_{z\in C} | F_k(z)| \leqslant K(1 + \epsilon)^k, \tag{14}$$

we obtain as above

$$|b_n| \leqslant K \sum_{k=n+1}^{\infty} \left(\frac{e\rho d^\rho(1+\epsilon)^\rho(\sigma+\epsilon)}{k}\right)^{k/\rho}$$

$$\leqslant K\left(\frac{e\rho d^\rho(1+\epsilon)^\rho(\sigma+\epsilon)}{n}\right)^{n/\rho}$$

for all sufficiently large n and therefore

$$\limsup_{n\to\infty} \frac{n}{e\rho}(E_n{}^2)^{\rho/n} \leqslant \sigma d^\rho$$

by virtue of lemma 2.

Conversely, suppose that (2) holds. By lemma 2, we will have

$$|b_n| \leqslant K\left(\frac{e\rho d^\rho(\sigma+\epsilon)}{n}\right)^{n/\rho}$$

and the representation (12) will hold for all z. Since, we also have, by (11), that

$$|p_n(z)| \leqslant K(n+1)^{1/2}|\varphi'(z)||\varphi(z)|^n$$

for all $z \in E$, that

$$|\varphi'(z)| \leqslant K$$

for all $z \in E$ (by (3)) and that, using (9),

$$|\varphi(z)| \leqslant \frac{|z|}{d-\epsilon}$$

for all z with sufficiently large modulus, we will have for those values of z that

$$|f(z)| \leqslant K\sum_{n=0}^{\infty}\left(\frac{e\rho d^\rho(\sigma+\epsilon)}{n}\right)^{n/\rho}(n+1)^{1/2}\left(\frac{|z|}{d-\epsilon}\right)^n$$

$$\leqslant K\sum_{n=0}^{\infty}\left(\frac{e\rho(\sigma+2\epsilon)}{n}\right)^{n/\rho}\left(\frac{d|z|}{d-\epsilon}\right)^n. \tag{15}$$

Since the largest term of the series $\sum_n (e\rho\lambda/n)^{n/\rho}x^n$ is $e^{\lambda x^\rho}$ and its value is attained for $n = e\rho\lambda$, inequality (15) implies that

$$|f(z)| \leqslant Ke^{(\sigma+3\epsilon)(d|z|/(d-\epsilon))^\rho},$$

that is, f is of type at most σ.

4. The Case $p > 2$

In view of the inequalities

$$E_n^2 \leqslant E_n^p \leqslant E_n^\infty \text{ for } 2 \leqslant p \leqslant \infty, \tag{16}$$

it is sufficient to consider the case $p = \infty$. We shall content ourselves with proving, say, (1) in that case. Suppose f is an entire function of finite order ρ. Then

$$\begin{aligned} E_n^\infty &\leqslant \max_{z \in C} \left| f(z) - \sum_{k=0}^{n} a_k F_k(z) \right| \\ &\leqslant \sum_{k=n+1}^{\infty} | a_k | \max_{z \in C} | F_k(z) |. \end{aligned}$$

Since, by lemma 1,

$$| a_n | \leqslant K n^{-n/(\rho+\epsilon)}$$

and since (14) holds, we obtain as above

$$\begin{aligned} E_n^\infty &\leqslant K \sum_{k=n+1}^{\infty} k^{-k/(\rho+\epsilon)} (1 + \epsilon)^k \\ &\leqslant K \sum_{k=n+1}^{\infty} \left(\frac{(1+\epsilon)^{\rho+\epsilon}}{n+1} \right)^{k/(\rho+\epsilon)} \\ &\leqslant K \left(\frac{(1+\epsilon)^{\rho+\epsilon}}{n} \right)^{n/(\rho+\epsilon)} \end{aligned}$$

that is

$$\frac{n \log n}{-\log E_n^\infty} \leqslant \frac{n \log n}{[n/(\rho + \epsilon)] \log n - \log K - n \log(1 + \epsilon)}$$

and

$$\limsup_{n \to \infty} \frac{n \log n}{-\log E_n^\infty} \leqslant \rho.$$

In view of inequalities (16) and the fact that (1) holds for $p = 2$, this last inequality actually is an equality. Finally, assuming (1) with $p = \infty$, we deduce from (16) that (1) will hold for $p = 2$ and hence that f is of finite order ρ.

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